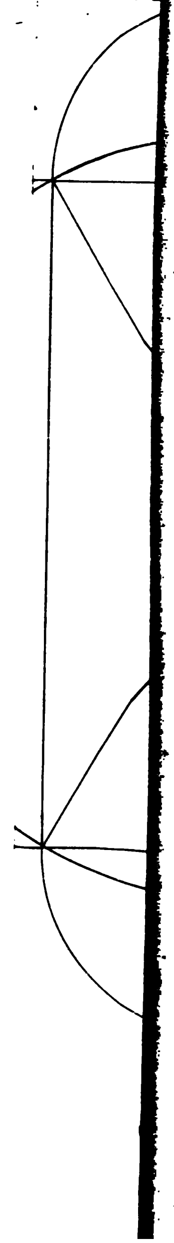
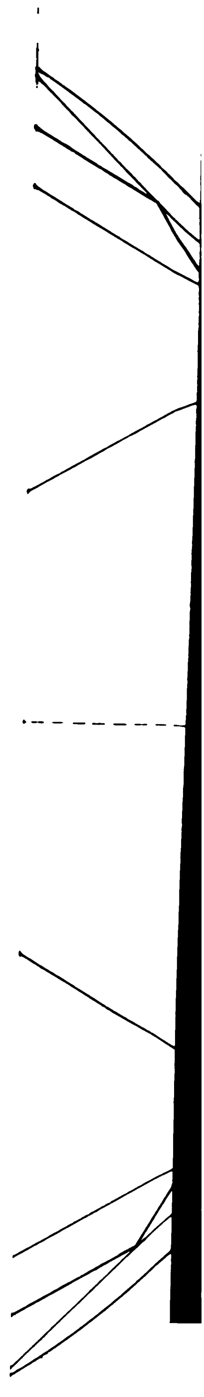
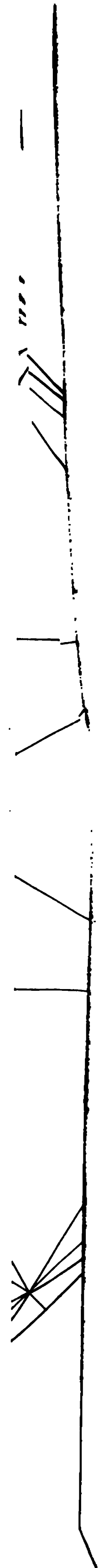
The Squared Circle

J. B. ANDREWS

FIG. 2







THE SQUARED CIRCLE:

BEING A SHORT TREATISE,
DESCRIBING THE MANNER BY WHICH ITS TRUE AREA
AND BOUNDARY WERE DISCOVERED.

BY
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THE SQUARED CIRCLE.

INTRODUCTORY.

IN the investigation of this problem I have avoided the usual method by which its solution has been hitherto attempted, viz., by increasing the number of the sides of a polygon enclosed within, and touching the boundary of a circle of any given diameter, as I consider that this mode of enquiry could never lead to any satisfactory or very certain results, even to the limited extent that they are *assumed* to be perfectly correct in their decimals, the absolute solution being admittedly unattainable by this mode of research. For when we take into consideration the great number and extent of the calculations required, and the great care necessary in their preparation in the forms of squares and square roots, to procure forty or fifty decimals perfectly correct—while the slightest error existing in any stage of the work must necessarily vitiate all that follows—we need not think it strange when we find that these decimals, even to the limited extent that they are assumed to be perfectly correct, are found differing from each other in the statements of different calculators. And who can be certain which of them, or if any one of them be correct, without proof?—a test which would require the reproduction of the whole work, which, say to obtain fifty decimals true, would require the most careful and incessant labour of a good calculator for some years; and even then, where is the absolute certainty of the results so found? With these considerations, I firmly believe that much valuable time and labour have been wasted in fruitless attempts to accomplish in this way what even to a limited extent is doubtful and uncertain; whilst the ascertainment of the absolute contents of a circle *by this means* must ever remain an absolute impossibility, so long as the *right line* forming a side of the polygon, however minute, remains unparallel with the *curved line* forming its circular boundary. And I speak from experience, as few persons have carried this mode of enquiry farther

than I have done myself; and in confirmation of my views, I would here notice that the above-stated mode was that adopted by the late Van Cullen in obtaining his well-known series of thirty-six figures, *assumed* to be *absolute* to that extent in giving the boundary of a circle of two inches diameter. I give the figures said to be engraved on his tomb at St. Peter's Church, Leyden, viz., 6·283185307179586476925286766559005 76. Yet we find these figures differing materially from those of his contemporaries. I need not give names, which could serve no purpose where all are differing.

And notwithstanding that I have since discovered a much more simple and certain mode of obtaining more perfect results with less than half the calculations required by the former method (which I hope to explain in the course of this enquiry), yet, as it is in some measure subject to the same objections as the former method, I do not intend pursuing my enquiry by it further, for the reasons already stated. And with my past experience, and a strong unwillingness to be defeated in obtaining the object of my research, I have again determined to pursue my enquiry by a new method altogether, and in this investigation I have directed my thoughts solely to the nature and *relative* value of certain circular rings, with a view to ascertain their *absolute* value, and that of the complete contents of the circle which they collectively constitute; and by this means I hope to show that I have been successful in obtaining all that can be reasonably desired or hoped for as a true and very simple solution of this very abstruse problem, as far as it relates to the circular *area.*

I have since been induced, however, to add some further very important proofs, procured from a distinctly different mode of research by *angular measurement.* I have also described a new mode of converting the angular boundary of the hexagon into a circular boundary, while it still retains the same linear extent, the area alone being thus proportionably increased; to all of which matters I have added, in the form of an Appendix, a short statement of my views regarding the nature of the errors existing in the present systems of circular and spherical geometry. I now beg to offer this little work—which has been the product of much time and mental labour—with the fullest confidence to the kind perusal and consideration of all that class of intelligent readers who would desire to see an improvement in that department of mathematical science to which the subject relates.

RESEARCH BY CIRCULAR RING MEASUREMENTS.

POSTULATES.

1. Let it be granted that a hexagon is composed of six equilateral triangles, two sides of each forming the radius, while their third or outer sides form the chords of the six arcs embraced within their surrounding circle.

2. And that an equilateral triangle of any dimensions is divisible into four equilateral triangles, the sides and height of each measuring one-half but containing only one-fourth the area of the larger one, so that a hexagon composed of six of the smaller triangles will only be equal to one-fourth the area of the larger hexagon.

3. And that the area of the six arcs occupying the spaces between the hexagon's six outer sides and their respective parts of the surrounding circle stand to each other in the same relative proportion as the triangles do which they cover; the larger arc, measuring twice the boundary and twice the height, is thus divisible into four parts, each of which is equal in area to that of the smaller arc.

4. And that if a circle be projected from the centre, and touching underneath the chords of the outer arcs, the curved triangles thus cut off the hexagons, and added to the arcs, will also contain four times the area of similar angles cut off the smaller hexagon; said triangles will also bear a certain relation to the hexagon from which they were separated, and also to the arcs, which their addition thus converts into a circular ring, to find the relative value of which to the entire circle I am thus conducted to my first hypothesis.

HYPOTHESES.

1. That the circular ring thus formed will be equal in area to that of a *circle* the *diameter* of which is exactly equal to the *chord of the outer arc.* (Proofs to follow—*see Figs. 3 and 4.*) Now, as the smaller circle has this *diameter,* consequently the outer circular ring should be exactly equal in area to the inner circle of 4 inches diameter, each thus representing in area one-fourth of the circle of 8 inches diameter, and the second circle three-fourths the value of same.

2. That the area of the six outer arcs is exactly equal to one-fifth of the area of the hexagon which they cover, or one-sixth the area of the entire circle. (Proofs to follow.)

3. That the total value or area of the curved triangles cut off the hexagon (by an inscribed circle), and which, by their addition thus, converts the six outer arcs into a circular ring, is exactly equal in area to one-tenth that of the hexagon, or to one-half that of the arcs. (Proofs to follow.)

Cor. 1. It would therefore follow, that if the hexagon be composed of 24 small triangles, and the six large arcs being also equal to 24 small arcs, the total value of which is equal to one-fifth that of the hexagon, consequently one small triangle and its fifth may be regarded as a true proportional constituting one of the 24 circular rings of which the entire circle is thus composed, or 6 triangles in same proportion will equal 5 circular rings, or 12 triangles will equal 10 circular rings, or 18 triangles will equal 15 circular rings, or 24 triangles will equal 20 circular rings, or $28\frac{4}{5}$ triangles will equal 24 circular rings, as constituting the whole circle of 8 inches diameter.

Or, by adding the value of one circular ring of $1\frac{1}{5}$ triangles, making a total of 30 triangles equal to 25 circular rings—thus also exhibiting a circle in five equal divisions.

In proof of my first hypothesis I would refer to *Fig. 1,* where the matter will be found simply established on the basis of the mathematical fact, that "in a right-angled triangle (ABC) the square which is described upon the side (AC) subtending the right angle is equal to the sum of the squares described upon the sides (AB and CB) which contain the right angle."—*Euclid,* prop. 47, B. 1.

The great principle of this beautiful proposition, with some modifications in the demonstrations, has also been found applicable as a mode of proving the relative proportions and areas of many other forms and magnitudes of regular figures bearing a similar relation to each other as that existing between the respective squares formed on the sides of the right-angled triangle of the figure. In confirmation of this statement, see Euclid's demonstration of prop. 31, B. 6—"If any similar rectilineal figures be similarly described on the sides of a right-angled triangle (ABC), the figure described on the side (AC) subtending the right angle is equal to the sum of the figures on the other sides."

Now, admitting the principle of the above two propositions as proved, I may confidently premise that the diameters of circles bear the same relative proportion to the areas of circles that squares formed on said diameters bear to the areas and the relative proportion of their respective squares. And, on the same principle, the areas of their semicircles will bear the same relative proportion to each other, when their respective diameters represent the three sides of a right-angled triangle, as that which half-squares bear to each other when constructed on the respective lines of the same figure.

For illustration I would refer to *Figs. 1, 2, 3, and 4,* where this right-angled triangle is described both singly and in duplicate, formed on half of a bisected equilateral triangle of one-sixth of the hexagon within a circle of 8 inches diameter, the hypotenuse equal to the radius of 4 inches or a square of 16 inches, and the small side of the right angle of 2 inches making a square of 4 inches, and the larger side of the right angle equal to the square root of a square of 12 inches, as given in calculation No. 4. I have also described the three circles *(Figs. 3 and 4)* separately, and it will be observed that I have projected the larger one of the two smaller circles from the common centre, and on the face of the *large circle (which is equal to both the smaller ones),* which thus exhibits a circular ring as a boundary in excess. Now it is manifest that this boundary must necessarily represent an area exactly equal to that of the small circle *(Fig. 4),* the small arc or segment which is common to both may be regarded as perfect in both (*Euclid,* ax. 8), while the *chord* of this segment must necessarily be equal to the *diameter of the small circle*—thus perfectly establishing my first hypothesis.

In proof of my second hypothesis I would refer to *Fig. 6,* representing a hexagon formed within a circle of 8 inches diameter, the arcs over each of its six sides having been bisected, and connected from this point with that terminating the chords of their respective arcs by twelve right lines, which will thus form the chords of the twelve arcs cut off the aforesaid six large arcs, and on adding their six enclosed triangles to the hexagon, we will thus form a polygon of twelve sides, enclosing an area of 48 inches—see *Fig. 7.* This is proved by multiplying the *radius* by a *side* of the *hexagon,* say 4×4 inches$=16$, one-half of which, say 8 inches, will give the area of each of the six figures, say $8 \times 6 = 48$ inches, per cal. No. 18; and I would here notice that the area

of this polygon is exactly equal to the area of a perfect square, formed by four times the square of the height of a large triangle of the hexagon, say 12 inches, as $12 \times 4 = 48$ inches; and that 12 inches is the absolute square of the *height* is easily proved by Euclid's 47 prop., B. 1, as it forms the larger side of the right-angled triangle, formed by a bisection of one large equilateral triangle of the hexagon. The hypotenuse will thus be formed by the 4-inch radius, = to square of 16 inches, from which deduct the square of the small side of the right angle, say 2 inches = to square of 4 inches: thus, $16 - 4 = 12$ absolutely, and 4 times $12 = 48$, and in another form of 3 squares, say $4 \times 4 = 16$, and 3×16 will also make 48 inches. I have also formed this square on the face of the circle (see *Fig. 7*), including an absolute area of 48 inches, or $\frac{3}{4}$ the area of a square of the diameter of the circle of 8 inches. I note this fact, as the area of this polygon and square respectively include the *height of the arcs* in combination with the hexagon; and as it is only the absolute area that I require, this saves me the necessity of following out this interminable square root farther than the 80 figures I have given in cal. No. 4. Now, on deducting the area of this polygon, per cal. No. 35, from that of the entire circle, per cal. No. 34, the area of the *12 outside arcs* will appear to be, per cal. No. 36, as the entire difference.

Fig. 6 I will now describe, and endeavour to explain the mode by which the area and boundary of our second polygon of 12 sides have been ascertained. First take the 12 arcs cut off the circle by the first polygon, the sides of which represent the chords of said arcs. We will bisect at a point each of these chords of the 12 arcs, and connect these points by 12 right lines, passing through and cutting off 12 triangles from the first polygon; and it will be noted that each of these 12 triangles represents exactly one-fourth the area of one of the 6 triangles from which the 12 outer arcs were separated. This will appear very evident, when we take into consideration that as the chords of the arcs, which also form the two upper sides of the triangles, are bisected at a point, the right lines connecting these points must each, therefore, equal one-half of its base in length, say half a side of the hexagon, or 2 inches; and that the upper portion of the triangle cut off must also be equal to one-fourth of its area, and at half the height of each angle. So that the second polygon thus produced, also of 12 sides, will not only cut off from the area of the

first polygon 12 of those angles, collectively equal in area to one-half of the 6 triangles under the outer arcs, but will also include the value of the other half within its own area, and in excess of that of the hexagon—see cals. Nos. 21, 22, and 23; while the *boundary extent* of *this polygon* and that of the *hexagon* remain exactly the same in their respective linear measurements, the duplication of the sides of *polygon No. 2* alone increasing the area, as shown in cal. No. 20. I will next direct attention to *Fig. 9*, where will be found the area of polygon No. 2, represented by 12 triangles, and also 6 squares, each square being of 4 inches area, all making together as per cal. No. 21, which, when compared with the area of 24 triangles, as per cal. No. 22, will show the same difference as stated per cal. No. 23.

I have also described 2 triangles on the face of each of the 6 squares, thus cutting off in excess a plain surface, the breadth of the square, say 2 inches, and in height equal to that of the arc of the triangle underneath, and the area of 6 of these magnitudes will form a quadrilateral figure, equal in area to the difference stated, as per cal. No. 23. This will appear by deducting from (24 inches) the value of the 6 squares the area of 12 triangles, per cals. Nos. 44 and 45, dif. 46. Now, to enable me to separate the value of this difference from the area of *polygon No. 2*, and thus to leave the 24 triangles separately and distinctly defined, and also to enable me to exhibit at same time the 18 triangles forming its outer series as absolutely constituting a simple circular ring, the height of a small triangle, and equal in area to $\frac{10}{16}$ or $\frac{15}{24}$ of the entire circle of 8 inches diameter, I will next describe a circle including $25\frac{1}{8}$ triangles, as per cal. No. 28. Now, on deducting the area of this circle from the *area* of the polygon No. 2, as given at cal. No. 27, we find the difference thus *cut off* its outer angles to be as per cal. No. 29. *(See Fig. 8.)*

Having now premised these facts, I would again call attention to the value of the angles cut off the polygon, per cal. No. 29; and as I have already shown, by cals. Nos. 21 and 22, that the area of cal. No. 23 represents the whole difference of polygon No. 2 in excess of that of the hexagon, and I have also shown, by cal. No. 46, that this amount is exactly contained in the area of the 6 plain magnitudes, marked off the squares, to show the excess of each square over the area of two triangles.

I would also call attention to cals. Nos. 24 and 25, showing what the items are of which cals. Nos. 23 and 26 are composed, from which we will find, that when the area of cal. No. 29 is cut off the

polygon No. 2, that there will only remain within it the area of one circular ring, as per cal. No. 25, in excess of the circular area of the hexagon. Now we shall find, by cal. No. 13, that this area is exactly represented by the 6 arcs of the inner circle as forming $\frac{1}{5}$ of the 6 triangles which they cover, and on separating the area of these 6 arcs from that of the 6 plain magnitudes cut off the squares, we will thus leave attached to the area of the 18 triangles above them 12 small triangular forms, value as per cal. No. 24, and exactly equivalent to that cut off the polygon No. 2, as per *Fig. 8* and per cal. No. 29; by which means you will plainly see that the exact area of the 18 triangles is perfectly preserved while being thus absolutely converted into a circular ring—the height of a triangle—and containing an area, as per cal. No. 31, equal to $\frac{10}{16}$ or $\frac{15}{24}$ of the entire circle.

And with a view to exhibit a still more perfect and complete illustration and proof of my second hypothesis, I purpose exhibiting the aforesaid 18 triangles, not only increased to the area of 18 circular rings, but also changed into the perfect form of 18 separate and distinct arched triangles—see *Fig. 9.*

First, in order to form them into 18 circular rings, it will be necessary to add $\frac{18}{5}$, or $3\frac{3}{5}$ triangles, to their collective area. Now, in order to do this, we will commence with the area of the 12 outside arcs, cut off the value of the circle by that of the first polygon, as per cal. No. 36, and next include the area cut off by that of the second polygon, as per cals. Nos. 20 or 37, and next the area of the angles cut off the latter polygon, as per cals. Nos. 29 or 38, which, when taken together, as appears per cals. Nos. 36, 37, and 38, will exactly make up the area of the $3\frac{3}{5}$ triangles, as per cal. No. 39. And it will be noted that each and all the items which make up this area lie outside of the 15 circular rings, and embrace all in that direction which the circular boundary encloses. Now, by cal. No. 39 we find that this amount is equivalent to 3 circular rings, or $\frac{3}{24}$ of the circular area, so that it may be added to the $\frac{15}{24}$ to make $\frac{18}{24}$; or it may also be formed into 18 separate arcs; so that, taking the whole series as they now stand to each other, from the first cal. No. $39=3\frac{3}{5}$ triangles, cal. No. $31=18$ triangles, cal. No. $32=1\frac{1}{5}$ triangles, as the value of the arcs of the 6 triangles represented by cal. No. 33, all together making, per cal. No. 34, $28\frac{4}{5}$ triangles, or 24 circular rings, as the entire contents of the circle of 8 inches diameter.

Now, in conclusion, I would simply call attention to the fact

that in forming the 18 outside arcs, it will be observed that, as only 12 of the 18 triangles are projected outward, the 6 turned inwards will require (owing to their curved form) one-half of the 48 small arcs of which the 12 large outside a·cs are composed, the area of each being equivalent to 4 of the small ones of the inner circle. I have shown how they may be applied on the face of the figure. But allowing the areas of those 12 arcs to remain in their position outside of the circle, we will thus have before us a perfect circular ring, solely composed of 18 triangles and their 18 arcs; the latter representing exactly $\frac{1}{5}$ of their area in addition, thus together making 18 circular rings, or $\frac{18}{24}$ of the entire circular area.

In proof of my third hypothesis, I think that it will be only necessary to refer to cal. No. 7, which represents the area of the 6 outside arcs as $\frac{1}{5}$, or $\frac{2}{10}$ of the hexagon, as per cal. No. 6. And, again, the same amount will appear represented by cal. No. 9 with half its value, equal to $\frac{1}{10}$ of the hexagon, added as per cal. No. 13, together making, as per cal. No. 11, $\frac{4}{16}$, or $\frac{6}{24}$ circular rings, containing the area of $7\frac{1}{5}$ triangles, and thus making $\frac{1}{4}$ of the entire circle. And to show that it is so, I would refer you to cal. No. 12, which represents $\frac{1}{4}$ the area of the hexagon, as per cal. No. 6, to which is added $\frac{1}{5}$ of that amount, as per cal. No. 13, making together, as per cal. No. 14, in like manner, $7\frac{1}{5}$ triangles, or $\frac{1}{4}$ the area of the entire circle, exactly equal to cal. No. 11. And these facts I presume will be regarded as a complete proof of my hypothesis No. 3.

The special object of my previous investigations being solely confined to the ascertainment of the true *area of a circle* of 8 inches diameter, by a process of *circular ring measurement*, without any reference to its boundary value, this being a natural consequent; say the area and diameter being given, to find the circumference, divide the area by one-fourth the diameter—the result will be the *circumference;* or the *circumference* and the *diameter* being given, to find the *area,* multiply the circumference by one-fourth the diameter—the result will give the area. And this matter, taken in connection with the various facts already described, I trust will be regarded not only as affording the most satisfactory proofs of my 1st, 2nd, and 3rd hypotheses, as found by the circular ring mode of enquiry, but also, at same time, serving to establish on the clearest evidence the true *area* and *boundary* of a circle of 8 inches diameter.

RESEARCH BY ANGULAR MEASUREMENTS.

I NOW purpose bringing forward some additional evidence in confirmation of that already given in relation to the values of areas and boundaries, the present proofs having been obtained by a distinctly different mode of enquiry from that of the former one, namely, from the *angular measurement* of the circular area; and the results so found will appear to be still further confirmed, or I should rather say completely established, by what I regard as the most important and interesting portion of all my investigations, namely, *the discovery of the true circular boundary,* simply obtained as the square root of a square of three figures.

I will now describe the mode by which I obtained both the results stated. First, to obtain the true area of the circle by angular measurement, to the *height* of the large triangle of the hexagon, as per cal. No. × 1, I added one-fifth of said height, as per cal. No. × 2, thus making together as per cal. No. × 3, and forming a perpendicular line extending from the centre to a short distance above the circular boundary—see *Fig. 10.* Now, when the height of this line is multiplied by 2 inches, or half the side of the large triangle (same as multiplying by 4 and taking half the area so found), it will thus give the true area of one *sixth* part of the polygon, as per cal. No. × 4; and this amount being further multiplied by 6, will give the true area of the entire circle, as per cal. No. × 5.

From this amount I will now deduct the area of the first polygon, as per cal. No. × 6, when there will then appear in the angular form, as occupying the space between the *boundaries* of the two *polygons,* exactly the *area* of the *12 arcs,* as per cal. No. × 7, and corresponding exactly to the extent of 80 figures with that already found by the circular ring measurement, as shown by cal. No. 36.

Yet I must here observe that the linear measurement of the angular boundary above stated, although found exactly capable of enclosing the true area of the circle, proves, nevertheless, to be considerably greater in extent than the *true circular boundary,* for, when measured as a boundary, it gives a greater area than it actually encloses.

This I have proved, by taking the square of 2, or half the base of the outer angle,$=4$, and to this square I have added the square of its height, say ·48, or $\frac{1}{25}$ of 12, that being the square of the height of the hexagon triangle; so from the square of 4·48 I have extracted the square root (which gives the hypotenuse of the right angle, and which also forms one side of this outer angle), and extended this root to the extent of 80 decimals. Now, 12 times the extent of this square root will give the total value of the *angular boundary* that encloses the true *area* of the circle, as given in cal. No. $\times$ 13.

Now, I think it is clearly manifest that no *angular* boundary of any *less dimensions* could possibly include the *same area,* as any *reduction* of the angular boundary must necessarily reduce the area that it encloses, as the angles, evidently, must cut off the circle exactly the very same area that each includes in the apex of its angle above the circle, so that a boundary line *less* than that of the angular one, which would also be capable of *enclosing the same area, must necessarily be the true circumference of the circle,* provided that it bears the test—viz., that half such boundary multiplied by half the diameter, or the whole boundary multiplied by one-fourth the diameter, will give the *true area as found by the angular measurement.*

Now, in order to find the boundary line required, I would here call your attention to the fact, that the angular measurement found in conformity with the 47 prop. of Euclid, B. 1, must necessarily include in the hypotenuse of each side of the six angles of which the whole boundary of the polygon is formed, say six squares of the height of the angle formed by and connecting its two sides, an excess the value of two squares, as the six angles collectively are in reality only equal to four right angles, and, consequently, the *areas* said angles enclose—see Euclid, prop. 32, B. 1 and cor. 7; so that, notwithstanding that the angular enclosure thus formed embraces the true *area* of the *circle,* yet its linear measurement, as a boundary, must therefoie be greater than that forming the true circumference of the circle, by the value of these two squares. In proof of this fact, I have deducted one-third from the square of the height of each of the six triangles, thus reducing the linear measurement of each hypotenuse forming each of their respective sides by that amount, and thereby embracing within the area of the six large angles collectively the value of four squares only, instead of six of their height.

Now, to bring out this calculation, I added to the *base* of the square of the right angle, say *4*, the square of its *height*, say ·48 − ·16, or one-third less, leaving ·32 as its reduced square; so from their sum, = 4·32, I extracted the square root to the extent of 80 figures, which gives exactly one-twelfth the linear boundary of the circle, as shown by cal. No. × 8. Now, by multiplying this amount by 6, and regarding it as linear measure only, we find that the product represents exactly one-half of the true boundary of the circle, as per cal. No. × 9.

Now, to prove this fact, we will next multiply this half boundary by half the diameter of the circle, say 4 inches, when we shall find its *true area* thus produced, as per cal. No. × 10, and constituting the very same amount as that given by the *angular* measurement for the *true area* of the circle, as per cal. No. × 5, so that we must therefore reasonably conclude that the square root of *4·32, as per cal. No. × 8,* multiplied by *12, as per cal. No. × 14, represents the true linear boundary of a circle of 8 inches diameter*, as no other form of boundary of this linear measurement, save the circular, could possibly enclose this area.

Now, before concluding this part of my investigation, I would desire to mention, that in the course of my previous investigation of this problem by the circular ring mode of enquiry, I had the pleasure of describing a very simple means of converting a series of 18 triangles into a perfect circular ring of $\frac{15}{24}$ of the true area of the circle, without effecting any change whatever in their collective area, or even in the depth of their circular ring. I would now, in like manner, wish to describe a very simple mode of converting the true angular form and measurement into the true circular form and measurement, both in boundary and area, so that I have no doubt that its very simplicity will be interesting. On looking back to the place where the mode of calculating the respective areas of the angular and circular boundaries is given, one important fact may have been observed as common to both—viz., that the height of the perpendicular line, formed by adding one-fifth to the height of the triangle, when multiplied by 2, gives one-sixth the true area of the circle; while, on the other hand, two-twelfths, or one-sixth of the circular boundary, when multiplied by 2, gives exactly the same amount. Now this fact can only be accounted for on one condition, namely, that the line represented by cal. No. × 3 must therefore be exactly equivalent to one-sixth of the true boundary of the circle, and on being compared with

cal. No. $\times 8$, it is found to be exactly one-half the extent given for one-twelfth of the circle—thus perfectly sustaining the fact stated.

So we thus find that, by simply converting the height of the angle in every sixth division of the polygon into the respective boundaries of each part, we thus at once produce the true circumference of the circle, as per cal. No. $\times 14$, and which, on being multiplied by 2, will also give its true area, as per cals. Nos. $\times 5$ and 10.

And I would also add, that this line, which represents extension only, without breadth, as given per cal. No. $\times 3$, when multiplied by one inch for area, will be exactly equal to one-tenth the area of the hexagon, as per cal. No. 6, and two of such lines will give the area of the six arcs of the hexagon; while one other of said lines will give exactly the area of the six angles required to be cut off the hexagon and added to said arcs in order to form a circular ring the height of the arc, which would thus represent one-fourth the area of the circle, thus leaving 9 times the area of said line to represent the remaining nine-tenths of the hexagon, or three-fourths the area of the circle; so that 12 times the area of said line, as per cal. No. $\times 3$, give the whole area of the circle, as per cal. No. $\times 5$, or 24 times the area of cal. No. $\times 8$, as being only half the area of cal. No. $\times 3$, will also give the same area. And I would here also notice, that by multiplying by 2 inches that portion of the height of said angle or line which appears (by *Fig. 10*) extended in excess of the height of the radius of 4 inches, as appears per cal. No. $\times 3$, it will give the area of two arcs of the 48-inch polygon, and 6 times that amount will give the area of the entire 12 arcs, as per cals. Nos. $\times 7$ and $\times 12$.

I now submit the foregoing results, found by *angular measurement*, as additional proofs of my 2nd hypothesis, with those already stated in page 11 as found by the circular ring mode of enquiry, while the matter contained in both, I trust, will be regarded as a most complete solution of the true area and boundary of a circle of 8 inches diameter.

I would here direct attention to *Fig. 5*, where will be seen *a square of 16 inches* enclosing an inscribed *circle of 4 inches diameter.* Now, in this case, the *linear measure* of the *boundaries* of *each* of these figures, when *multiplied by 1*, or one-fourth the *diameter* of the circle, will exactly represent their respective areas, say the square $= 16$, and the circle *boundary* and also its *area* each $=$ cal. No. $\times 9$. I would also notice *Fig. 5a* as a mode of dividing the circle into four parts.

THE ANGULAR HEXAGON BOUNDARY CONVERTED INTO CIRCULAR.

I WOULD next desire to describe what I regard as not the least important part of all my investigations, as it exhibits a very simple mode by which the area of any hexagon can be absolutely increased, not only into that of a perfect circle, formed from the hexagon by the addition of one-fifth, as I have already shown, but also into the area of a perfect circle, the *circumference of which* shall possess *the same linear measure* as that of the *hexagon;* and also to describe a very simple mode of finding the *true diameter of said circle.*

I was conducted to this solution as the result of my efforts to follow out a continued series of duplication (to a large extent) of the sides of each of the successive polygons, generated from the first polygon of 12 sides, which has an area of 48 inches, and by which process the linear measure of the sides *collectively* of each polygon remains the same as that of the hexagon throughout every successive stage of the duplication of their sides, as I considered that this mode of reducing the sides of a polygon, while approaching the circular area, was much more perfect for that purpose, and not requiring half the amount of the calculations in the form of squares and square roots, and not being subject to certain defects of the old method, whereby the operator may be himself deceived in the production of a *circumscribed* polygon, which he only intended to be an *inscribed* polygon—see Appendix. The first stage of the process, by which the duplication of the sides of the hexagon has been accomplished, will be found in the manner by which the second polygon was formed, as described in the latter part of pages 8 and 9 (see *Fig. 6* and cal. No. 19), which is perfectly similar to the way by which all the subsequent stages of the duplication of the polygon sides are effected; so, to make the process very plain, I think it will be only necessary to describe how the next duplication of the sides of polygon No. 2 should be done—viz., first, form a circumscribing circle around the polygon, and bisect at a point each of the arcs thus formed above its respective sides; next, form

within the area of each arc a triangle, having for its base the side of the polygon, and for its apex the point of bisection in the arc; next, bisect at a point each of the lines which form the upper sides of each triangle, and by simply connecting these points by right lines, you will form the 24 sides of the next polygon, now only of 1 inch each, so that while the polygon increases in area by the value of half its angles, as at the first division, the *collective* surface value of their sides will remain the same as that of the hexagon, through every subsequent stage of their duplication by a similar process, till they reach the most minute stage of their subdivision; and well knowing that so long as the *side* of the polygon possessed numerical value, it must necessarily form the *chord* of the *arc* formed by the next circumscribing circle, the greater linear extent of which it could never reach, and consequently could never as a polygon conform to the conditions required for a perfect circle—viz., "*A circle is* a plane figure *bounded by one line*, and is such that *all right lines drawn from its centre to the circumference are equal to one another.*"—*Euclid.*

Now, under this difficulty, after many trials, I found that the total sum of the additions to the area of the hexagon, caused by the duplication of the polygon sides (to a limited extent), approached so closely to a ninth of its area, that I tried a ninth part added to the area of the hexagon, which I happily found gave the *perfect area of a circle* having for its circumference a linear extent of 24 inches, equivalent to that of the hexagon boundary; and this I have shown, per cals. Nos. $\times 26$, $\times 27$, and $\times 28$, to be perfectly correct to the extent of 80 decimals. I have also proved from it the true proportional which the circumference bears to the area, and which both bear to the diameter.

First, the true *diameter* is found by dividing the circular *area* by *one-fourth of its boundary*, say 6 inches—results as per cal. No. $\times 29$; and next, by multiplying the boundary by one-fourth of said diameter, per cal. No. $\times 30$, or (what will give the same results) by multiplying one-fourth the diameter by the boundary of 24 inches, say by 4 and by 6, which will again give the *true area* of the circle. It will here be noticed that the last calculation in this series also exhibits the fact that *6 times* the *diameter*, per cal. No. $\times 31$, has thus also produced the area of the circle, as per cal. No. $\times 32$, and which exactly corresponds with the amount produced by adding a ninth part to the area of the hexagon, as per cal. No. $\times 28$.

And I have further exhibited the correctness of these calculations

by subtracting one-ninth part of the area of the hexagon of 4 inch sides, as per cal. No. × 34, from one-fifth of the area of said hexagon, as per cal. No. × 33; from which we find that the difference thus appearing, as per cal. No. × 35, exactly corresponds with that existing between the respective areas of the large and the smaller circle, as appears per cal. No. × 38. Now, from the facts already stated, the area of a circle of 24 inches circumference may be regarded as being composed of ten parts, nine of which (in the form of a polygon) represent the area of the hexagon, while the tenth part thus absolutely represents the area of the arcs of said polygon, which thus transforms the whole into the form and area of a perfect circle, so that we thus find that the *circular form* of boundary embraces a *tenth more* in *area* than that of the *hexagon form* possessing the same extent of boundary. And I would also notice that the area of this circle may also be divided into *six parts,* five of which will occupy the area of its own hexagon, and the remaining sixth part that of the six arcs, each side of the hexagon being equal in extent to one-half the diameter of the circle, as per cal. No. × 29, and will also represent a true proportional in all its parts, if compared with each corresponding part of the large circle.

I have still many interesting facts to bring forward in relation to this subject, but I find time and advancing years admonish me to bring this very arduous and wearying investigation to a conclusion; and especially as I consider that no further evidence can be reasonably required to fully establish my claim to the discovery of the perfect mode of finding the *true area of the circle,* and also *its true circumference and diameter,* whereby this long-vexed problem may be finally set at rest by its perfect and very simple solution.

APPENDIX.

BEFORE finally retiring from this investigation, I would desire to make some brief observations, with a view to explain what I believe would account for the very important difference existing between the linear measurement of the true circular boundary which I have just described, when compared with that obtained by the continued duplication of the sides of a polygon carried to a great degree of minuteness, supposed to be formed within the boundary of a circle of a given diameter. The external surface measurement of a polygon thus formed, being regarded as the nearest approximation to the true circular boundary, has been thus adopted in all our systems of circular and spherical geometry.

Now, before directly entering into an explanation of the causes which operate to produce the difference in the latter boundary, I would desire to lead to such explanation by first pointing out that I have already shown that an angular boundary, however regular may be its form, or minute its polygon sides, should such boundary be capable of embracing the same area as that of a given circular one, it must necessarily be very much in excess of the circular one in its linear extent, and so much so, that it could not possibly be formed within the true circular enclosure; and it is also manifest that should we try to measure the collective surfaces of all these angular or minute polygon sides as a boundary, in such case the full extent of this surface measure, when thus erroneously valued as a circumference, would necessarily give as its area that of a *circle*, the *boundary* of which would be equal to that of the angular one. The error of doing so will appear very manifest on looking at the linear extent of the angular boundary, as given at cal. No. $\times$ 13, compared with that of the true circular one, as given at cal. No. $\times$ 14 (both enclosing the same area); the excess in extent of the former over the latter also appears by cal. No. $\times$ 15. It is,

therefore, very evident that, should we measure the extent of No. ×13 as a boundary, it would thus be treated as a larger circular boundary than No. ×14 to the extent of No. ×15, and would thus give an area of *more than twice* the extent of cal. No. ×15, multiplied by one, in *excess* of *what its angular boundary actually contains.* For if half the boundary multiplied by half the diameter gives the true area of a circle, consequently the whole boundary multiplied by one-fourth the diameter will give the same amount. But in this case the *diameter* would also be enlarged in proportion to the greater extent of the boundary; to find the value of this increase as a proportional, say as the true boundary, per cal. No. ×14, is to the diameter of 8 inches, so is the boundary of cal. No. ×13 to the diameter required for it; and one-fourth the extent of the diameter thus found, when multiplied on cal. No. ×13, will give the amount of its supposed circular area; and this being so greatly in excess of the area which it actually encloses, clearly proves the utter absurdity of measuring it as a boundary to find its true area.

Having premised these facts, I will now proceed with my investigation as to the merits of the afore-mentioned polygon boundary, supposed to have been obtained from the linear measure of the angular surfaces of a numerously-sided polygon, assumed to be constructed *within* the area of the true circular boundary, so that each minute side of said polygon is thus regarded as forming a separate and distinct chord of a corresponding arc or segment of their enclosing boundary. But the *area* of those *segments*, and the *extra boundary measure* of their *curvature*, is regarded as being cut off, or at least not included in either the boundary or area of the polygon measurement. Yet it must be admitted as an axiom, that both the *area and boundary* of the enclosed polygon must necessarily be *less* than that of its surrounding circle, which *includes* those segments.

Now, the first thing to ascertain is the simple fact—has the polygon, as stated, been actually formed *within the area* of the true circular boundary? This is very easily found by taking, say, Van Cullen's boundary of the circle for a diameter of 2 inches, which, on being multiplied by 4, will give the proportional boundary for a diameter of 8 inches; and comparing this boundary with that given as the true boundary for 8 inches diameter, per cal. No. ×14, I have chosen to compare this with the Van Cullen boundary, although I am aware that the

ten-decimal product of the inscribed and circumscribed polygons is that which is generally adopted in works on circular and spherical geometry. But as the Van Cullen boundary of 35 decimals agrees with the former to the extent of 9 of the said 10 decimals, it will thus attest the correctness of both to that extent, which also represents the greater part of all the difference.

CIRCULAR BOUNDARIES.

Van Cullen's polygon boundary...	=25·132741228	71834590770114706623602304	for 8 in. dia.
True circular boundary, ℘ No. × 14	=24·941531628	99183302679522731768456208	„ „
Difference in excess	= ·191209599	72651288090591974855146096	

It is therefore manifest that the Van Cullen polygon *was not constructed within the true circular boundary;* as, in such case, the chords formed by its sides could not possibly have extended beyond their circular enclosure, nor be even equal to it in their linear extent. But, on the contrary, we find them extending considerably beyond it, and even including a much larger area than the circle itself, as appears by the above figures, that is, if they be regarded or measured for *area* as a *circular boundary.* For, as the true boundary for 8 inches diameter when multiplied by 2 inches, or one-fourth the diameter, will give the *true area,* consequently the above boundary, when multiplied by one-fourth of a *diameter proportional thereto, would give an area* considerably greater even than the above figures multiplied by 2 would yield, and the excess would also show more than twice the amount above stated.

MEASURE OF THE AREAS OF CIRCLES.

V. C.'s supposed area of circle	=50·265482457	43669181540229413247204608	for 8 in. diameter.
True circular area	=49·883063257	98366605359045463536912416	„ „
Difference in excess	= ·382419199	45302576181183949710292192	

I have also examined the *area* of the inscribed polygon of 10 decimals, as given in "Leslie's Geometry," page 355, for an assumed radius of 1 inch, or half the area of a circle of 2 inches diameter. And having increased this area to a proportional for 8 inches diameter, and taking one-half of the linear extent of the figures representing that amount for the measure of the outer sides of the polygon, I then divided this extent by the given number of sides to obtain the measure of a single side, which, on being multiplied by *the height of the triangle*

underneath valued as 4 inches, and taking half the amount for its area, gave exactly a true proportional of the value given for the *whole area of the polygon as stated*, which, had it been multiplied by any *less* amount, it could not have done. It therefore follows that the height given for the common radius of the inscribed and circumscribed circles, which is stated to be *considerably less* than the *height* of the centre of a *side* of the inscribed *polygon*, cannot be correct as a proportional of the area given for that of its total contents. It thus appears very evident that it would require a radius *not less* than 4 inches to form an inscribed circle only touching the centres of the respective sides of a circumscribing polygon of the area given, as clearly appears by the figures stated on page 21; and the value of the *angles thus cut off* the area of the polygon by the inscribed circle is also given in the figures just referred to, as so much in *excess of the true circular area*. Now, without pursuing this enquiry farther, I think what I have already brought forward fully justifies me in stating that neither the surface measure of the circumscribing polygon nor its area can be regarded as a true proportional of the diameter given, which only applies to a perfect circle, and not to the polygon. In confirmation of these facts, I think that it is only necessary to give "Euclid's" definition of what constitutes a circle and its diameter—"A circle is a plane figure bounded by *one line*, which is called the circumference; and is such, that all right lines drawn from a certain point within the figure to the circumference are *equal to one another*, and this point is called the *centre* of the circle. A diameter of a circle is a right line drawn through the centre, and terminated both ways by the circumference."—Defs. 15, 16, and 17, B. 1.

Now, a *polygon* is not bounded by *one line*, nor are all right lines drawn from its centre to the circumference equal to one another; not being equidistant from the centre they cannot be, and consequently the *true circle diameter*, as described, cannot be applied to the *arca* of a *polygon, nor the boundary of the latter to that of the perfect circle.*

In conclusion, I would simply note that I consider the similarity in value of the boundaries and areas appearing in both polygons merely indicates the idea that their respective authors were equally deceived by the mode adopted by each to obtain their figures; while entertaining the supposition that they could actually place within the six arcs formed under a circumscribing circle around a hexagon nearly the entire area of six squares, within a space only capable of containing *four* of like

magnitude. This appears in the *excess* of their *boundaries* and *areas* over that of the true circle, which represents nearly one-third of the area of their included squares, evidently so much in excess of what they were *intended* to occupy. The explanation of the errors stated will appear very plain on reference to page 13, where may be seen my reasoning, and the mode explained by which I was enabled to obtain the true circular boundary.

The mode of reckoning by the common principles of arithmetical computation adopted in this little work, has been specially intended to render the enquiry more simple for the perusal and consideration of the large portion of intelligent people to whom this form of investigation will be familiar. And I have carried out the work so as to save trouble to the reader, as far as possible. All the calculations referred to in the book will be found already prepared with the most perfect accuracy; even the *squares* are given, and their roots extended in the circular ring part to 80 and in the angular part to 82 decimals. I have also given the diagrams or figures at their full size, that they may more perfectly speak to the eye, while the calculations demonstrate the facts to the understanding.

In conclusion, I would now, as an Irishman of the Province of Ulster, desire to submit the result of my investigations (as a small contribution to geometrical science) to my country, and the great nation of which I am proud to be a member; and also to scientific societies generally, soliciting their kind and careful examination of the same.

Marcus Ward & Co., Limited, Printers, Belfast.

-CALCULATIONS.

NCHES DIAMETER, AND ITS PROPORTIONALS,
ULAR RING MEASUREMENTS.

366942805253810380628055806979451933016908800
4733885610507620761256111613958903866033817 60

NDARIES OF CIRCLE 8 INCHES DIAMETER.

Enclose 086818488157591521731536010808330260703010 72280
" 083976395654869481044003620504107835443486 72048

002842092502722040687532390304222425259524 00232

Enclose 086818488157591521731536010808330260703010 72280
" 521510614526713329326144307902245469575884 99332

565307873630878192405391702906084791127125 72948

True Ci 083976395654869481044003620504107835443486 72048
Inscribe 521510614526713329326144307902245469575884 99332

562465781128156151717859312601862365867601 72716

Linear e 840331366304572456753666968375342319620290 56004
" 626792551210559444110512025658520455797990 41611

213538815094013012643154942716821863822300 14393

AGON CONVERTED INTO CIRCULAR BOUNDARY.

806627326091449135073339367506846392405811 2008
645180814010161015008148818611871821378423466 7$\frac{5}{9}$

451808140101610150081488186118718213784234667 5$\frac{5}{9}$

741968023350268358346914697686453035630705777 9$\frac{7}{27}$

185492005837567089586728674421613258907676444 4$\frac{22}{27}$
× 4

741968023350268358346914697686453035630705777 9$\frac{7}{27}$
× 6

451808140101610150081488186118718213784234667 5$\frac{5}{9}$

361325465218289827014667873501369278481162240176
645180814010161015008148818611871821378423466755

716144651208128812006519054889497457102738773421

167952791309738962088007241008215670886973440976
451808140101610150081488186118718213784234667555

716144651208128812006519054889497457102738773421